Exercise 5

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 4y' + 5y = e^{-x}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 4y_c' + 5y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 4(re^{rx}) + 5(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 4r + 5 = 0$$

Solve for r.

$$r = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$r = \{2 - i, 2 + i\}$$

Two solutions to the ODE are $e^{(2-i)x}$ and $e^{(2+i)x}$. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{(2-i)x} + C_2 e^{(2+i)x} \\ &= C_1 e^{2x} e^{-ix} + C_2 e^{2x} e^{ix} \\ &= e^{2x}(C_1 e^{-ix} + C_2 e^{ix}) \\ &= e^{2x}[C_1(\cos x - i \sin x) + C_2(\cos x + i \sin x)] \\ &= e^{2x}[(C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x] \\ &= e^{2x}(C_3 \cos x + C_4 \sin x). \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 4y_p' + 5y_p = e^{-x} \tag{2}$$

Since the inhomogeneous term is an exponential function, the particular solution is $y_p = Ae^{-x}$.

$$y_p = Ae^{-x} \quad \rightarrow \quad y_p' = -Ae^{-x} \quad \rightarrow \quad y_p'' = Ae^{-x}$$

Substitute these formulas into equation (2).

$$Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x}$$

$$10Ae^{-x} = e^{-x}$$

Match the coefficients on both sides to get an equation for A.

$$10A = 1$$

Solving it yields

$$A = \frac{1}{10},$$

which means the particular solution is

$$y_p = \frac{1}{10}e^{-x}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= e^{2x}(C_3 \cos x + C_4 \sin x) + \frac{1}{10}e^{-x}, \end{aligned}$$

where C_3 and C_4 are arbitrary constants.